Rover Publications

United International Journal of Engineering and Sciences (UIJES)

An International Peer-Reviewed (Refereed) Engineering and Science Journal Impact Factor:7.984(SJIF)
Volume-4, Special Issue-3; ISSN: :2582-5887

Additive Digital Groups, Rings, Fields and Vector Space

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DOI:10.53414/UIJES:2024.43.525

***Abstract* – In this paper we introduce – Additive digital group, digital ring, digital field, digital vectors space.**

***Keywords* – Set, Group, ring, field, Vector Space, additive digital groups., digital ring, digital field, digital vectors space.**

I. Introduction

In modern algebra the study of groups, which are systems. Consisting of set of elements and a binary operation that can be applied to two elements of the set which together satisfy certain axioms. These require that the group be closed under the operation (the combination of any two elements) produces another element of the group that it obey the associative law that it contain an identity element (which combined with any other element leave the later unchanged) and that each element have an inverse (which combines with an element to produce the identity element). If the group also satisfies the commutative law it is called a commutative or abelian group. The set of integers under addition, where the identity element is "0" and the inverse is the negative of a positive number or vice versa is an abelian group.

A Ring is a set equipped with two operations, called addition and multiplication. A Ring is a Group under addition and satisfies some of the properties of a group for multiplication. A field is a group under both addition and multiplication and satisfies distributive properties.

In mathematics and physics, a vector space is a set whose elements often called vector may be added together and multiplied by numbers are called scalars. Scalars are often real numbers but can be complex numbers or more generally elements of any field.

We introduced the digital group, digital ring, digital field, digital vector space, digital discrete topological group, digital group constructed with only two elements "0" and 1.

II. Preliminaries

In this section we given some definitions and state some results for later use.

Definition 2.1:- If G is a non empty set and o is a binary operation defined on G such that the following laws are satisfied (G, o) is a group.

G_1 : Associative law: For a, b, c ∈ G, (a o b) o c = a o (b o c)

G_2 : Identity law : ∃ e ∈ Gsuch that aoe = a= eoa for every a∈ G is called an element in G.

G_3 : Inverse law: For each a∈ G ∃ an element b ∈ G such that aob = boa = e :b is called an inverse of a

Example 2.1 :-The set of six transformations $f_1, f_2, f_3, f_4, f_5, f_6$ on the set A = c - {0,1} defined by forms a finite group of order 6 with respect to composition of functions as the composition

$f_1(z) = z$, $f_2(z) = \frac{1}{z}$, $f_3(z) = 1-z$, $f_4(z) = \frac{z}{z-1}$, $f_5(z) = \frac{1}{1-z}$, $f_6(z) = \frac{z-1}{z}$ forms a finite group of order six w.r.t composition of functions as the composition.

Let G = { $f_1, f_2, f_3, f_4, f_5, f_6$ }, f_i : A →A, Since ∀z ∈ A, $f_1(z) = z$, f_1 is the identity function

0	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6

f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

Example 2.2 : Real quaternior group. Let T = $\{a_0 + a_1 i + a_2 j + a_3 k \,/\, a_0, a_1, a_2, a_3 \in R\}$ where I,j,k are such that $i^2 = j^2 = k^2 = -1$
ij =-ji =k, jk = -kj = i, ki = -ik = j and ijk = -1
Also $a_0 + a_1 i + a_2 j + a_3 k = b_0 + b_1 i + b_2 j + b_3 k \Leftrightarrow a_0 = b_0, a_1 = b_1, a_2 = b_2, a_3 = b_3$
Define an operator $\oplus$: T x T $\longrightarrow$ as follows
$(a_0 + a_1 i + a_2 j + a_3 k) \oplus (b_0 + b_1 i + b_2 j + b_3 k)$
$= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$
then (T, $\oplus$) is a group

Definition 2.2 : Let R be a non empty set + , * be two binary operations in R. (R, +,*I) is said to be a ring if, for a, b, c ∈ R
R_1 a+b = b+a
R_2 (a+b) + c = a+(b+c)
R_3 there exists o∈ R such that a+o = a for a∈ R
R_4 there exists -a∈ R such that a+(-a) = 0 for a∈ R
R_5 (a.b).c = a.(b c) and
R_6 a(b+c) = a . b + a . c and (b+c).a = ba+ca

Example 2.3 :-The set $R = \{a, b\}$with addition '+' and multiplication ' *'defined as followsis a ring.

+	a	b
a	a	b
b	b	a

*	a	b
a	a	a
b	a	b

Example 2.4 :- The set z (i) = { a + bi / a,b ∈ z, $i^2 = -1$ } of Gaussian integers is a ring with respect to addition and multiplication of numbers

Example 2.5 : M_2 (z) be the ring of all 2 x 2 matrices over z

<u>Definition 2.3 :-</u> Let F be a non empty set and + and. Are binary operations on F then the algetoria system $(F, +, .)$ is said to be a field if.
(i) $(F, +)$ is an abelian group
(ii) $(F, .)$ is an abelian group
(iii) Distributive Laws.

Example 2.6 : The set Q = R x R x R x R = $\{\alpha_0, \alpha_1 i + \alpha_2 j + \alpha_3 k \,/\, \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in R\}$ where i,j,k are qua units is a field.

Example 2.7 : The set 2 x 2 matrices of the form $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ is a field for compositions of matrix addition and multiplication

Definition 2.4 :
Let $(F, +.)$ be a field. The elements of F will be called scalars. Let V *be* a non empty set whose elements will be called vectors. Then V is said to be a vector space over the field F, if
1. These is defined an internal composition in V called addition of vectors and denoted by '+' also for this composition V is an abelian group.
2. There is an external composition in V over F called scalar multiplication and denoted multiplicatively i.e., $a\,\alpha \in v$ for all $a \in F$ *and* $\alpha \in V$.
3. The two compositions, scalar multiplication and addition of vectors satisfy the following particulars.

$$\left.\begin{array}{l}(i)\quad a\,(\alpha+\beta)=a\alpha+b\beta\\(ii)(a+b)\alpha=a\alpha+b\alpha\\(iii)(ab)\alpha=a(b\alpha))\\(iv)\ \ 1.\alpha\ =\ \alpha\end{array}\right\}\quad \forall\, a,b\ \in F\ \ and\ \ \alpha,\ \beta\ \ \epsilon\, v$$

Example 2.8: The set of all convergent sequence is a vector space over the field of real numbers.

Example 2.9: The set of all vectors in a plane over the field of real numbers is a vector space.

Example 2.10: The set of all polynomials of degree ≤ n over F can be made into a vector space over F

III. ADDITIVE DIGITAL GROUP

In this section we introduce additive digital group, digital ring, digital field, digital vector space

Definition 3.1.1 :- If G is a non empty set and o is a binary operation defined on G such that the following laws are satisfied (G, o) is a group.
G_1 : Associative law: For a, b, c ∈ G, (a o b) o c = a o (b o c)
G_2 : Identity law : ∃ e ∈ Gsuch that aoe = a= eoa for every a∈ G is called an element in G.
G_3 : Inverse law: For each a∈ G ∃ an element b ∈ G such that aob = boa = e :b is called an inverse of a

Example 3.1.1 : Let $G=\{0,1\}$ the operation is $+_2$

$+_2$	0	1
0	0	1
1	1	0

Clearly $(G,+_2)$ is digital group w.r.t. to binary digital addition

Example 3.1.2 Let $G=\{00,\ 01,\ 10,\ 11\}$ $w.r.t.$ binary addition $+_2$

$+_2$	00	01	10	11
00	00	01	10	11
01	01	10	11	00
10	10	11	00	01
11	11	00	01	10

Clearly $(G,\ +_2)$ is digital group w.r.t. to binary digital addition

Example 3.1.3 : If we take binary addition as binary composition.
Then $G=\{000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111\}$ is additive digital group.

$+_2$	000	001	010	011	100	101	110	111
000	000	001	010	011	100	101	110	111
001	001	010	011	100	101	110	111	000
010	010	011	100	101	110	111	000	001
011	011	100	101	110	111	000	001	010
100	100	101	110	111	000	001	010	011
101	101	110	111	000	001	010	011	100
110	110	111	000	001	010	011	100	101
111	111	000	001	010	011	100	101	110

Clearly $(G\,+_2)$ is digital group w.r.t. to binary digital addition
Note: In the above table we left the left-side last digit [11].

Example 3.1.4: The multiplicative digital group
S_n = Perm (n) which consists of the permutation n-matrices: n x n matrices having exactly a single 1 in each row and column, and otherwise entries of 0. There are n! such matrices. For the low dimensional cases we have
S_1 = {(1)}

$S_2 = \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right\}$

$S_3 = \left\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}\right\}$

IV. DIGITAL RING

We introduce digital ring which is a Ring with binary addition and multiplication it satisfies all the properties of the usual ring .

Definition 3.2.1 : : Let R be a non empty set + , * be two binary operations in R. (R, +,*I) is said to be a ring if, for a, b, c ∈ R
R_1 a+b = b+a
R_2 (a+b) + c = a+(b+c)
R_3 there exists o∈ R such that a+o = a for a∈ R
R_4 there exists -a∈ R such that a+(-a) = 0 for a∈ R
R_5 (a.b)*c = a*(b c) and
R_6 a(b+c) = a * b + a * c and (b+c).a = ba+ca

Example 3.2.1: $R = \{0, 1\} = \{R, +_2, *_2\}$ binary addition and binary multiplication

$*_2$	0	
0	0	1
1	1	0

$*_2$	0	1
0	0	0
1	0	1

Clearly $(R, +_2, *)$ is digital ring w.r.t. to binary digital addition

V. DIGITAL FIELD

Definition 3.3.1: Let F be a non empty set and + and. Are binary operations on F then the algetoria system $(F, +, .)$ is said to be a field if.
(i) $(F, +)$ is an abelian group
(ii) $(F, .)$ is an abelian group
(iii) Distributive Laws.

Example 3.3.1: If DF= $\{0, 1\} = \{DF, +_2, O_2\}$ binary addition and binary multiplication

$+_2$	0	1
0	0	1
1	1	0

$\cdot_2$	0	1
0	0	0
1	0	1

$(DF, +_2, \cdot_2)$ is a field

VI. DIGITAL VECTOR SPACE

Definition: 3.4.1 : *Let* $(F, +.)$ be a field. The elements of F will be called scalars. Let V *be* a non empty set whose elements will be called vectors. Then V is said to be a vector space over the field F, if
1. These is defined an internal composition in V called addition of vectors and denoted by '+' also for this composition V is an abelian group.

2. There is an external composition in V over F called scalar multiplication and denoted multiplicatively i.e., $a\,\alpha \epsilon\, v$ for all $a \in F$ and $\alpha \epsilon V$.
3. The two compositions, scalar multiplication and addition of vectors satisfy the following particulars.

$$\left.\begin{array}{l}(i)\quad a\,(\alpha+\beta)=a\alpha+b\beta \\ (ii)(a+b)\alpha=a\alpha+b\alpha \\ (iii)(ab)\alpha=a(b\alpha)) \\ (iv)\ \ 1.\alpha\ =\ \alpha\end{array}\right\}\ \forall\, a,b\ \in F\ and\ \alpha,\ \beta\ \ \epsilon\, v$$

$Example$ 3.4.1 $V = \{0,.1\}$ one dimensional vector space over the field F = {0, 1} under $+_2$ and 0_2

Vector addition

$+_2$	0	1
0	0	1
1	1	0

and

Scalar Multiplication

1(0) = 0
1(1) = 1
0(1) = 0
0(0) = 0

$Example$ 3.4.2 : Let $F = \{00,\ 01,\ 10,\ 11\}$ $w.r.t.$ binary addition

$+_2$	00	01	10	11
00	00	01	10	11
01	01	10	11	00
10	10	11	00	01
11	11	00	01	10

Clearly $(V,\ +_3)$ is a Digital vector space

Example 3.4.3 : If we take binary addition as binary composition.
Then F= $\{000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111\}$ is additive digital group.

$+_2$	000	001	010	011	100	101	110	111
000	000	001	010	011	100	101	110	111
001	001	010	011	100	101	110	111	000
010	010	011	100	101	110	111	000	001
011	011	100	101	110	111	000	001	010
100	100	101	110	111	000	001	010	011
101	101	110	111	000	001	010	011	100
110	110	111	000	001	010	011	100	101
111	111	000	001	010	011	100	101	110

Clearly $(V\,+_2)$ is an additive digital Vector space

VI. CONCLUSION

In this paper the concept of digital group, digital ring, digital field and. digital vector space are introduced. It is hoped that these concepts will rise to the notions like digital ideals, digital modules, digital polynomial rings etc.

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